A note on module-composed graphs

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Abstract

In this paper we consider module-composed graphs, i.e. graphs which can be defined by a sequence of one-vertex insertions $v_1, \ldots, v_n$, such that the neighbourhood of vertex v_i, $2 \leq i \leq n$, forms a module (a homogeneous set) of the graph defined by vertices $v_1, \ldots, v_{i-1}$.

We show that module-composed graphs are HHDS-free and thus homogeneously orderable, weakly chordal, and perfect. Every bipartite distance hereditary graph, every (co-$2C_4, P_4$)-free graph and thus every trivially perfect graph is module-composed. We give an $O(|V_G| \cdot (|V_G| + |E_G|))$ time algorithm to decide whether a given graph G is module-composed and construct a corresponding module-sequence.

For the case of bipartite graphs, module-composed graphs are exactly distance hereditary graphs, which implies simple linear time algorithms for their recognition and construction of a corresponding module-sequence.

Keywords: graph algorithms, homogeneous sets, HHD-free graphs, distance hereditary graphs, bipartite graphs

1 Preliminaries

Let $G = (V_G, E_G)$ be a graph. For some vertex $v \in V_G$ we denote the *neighbourhood* of v by $N(v) = \{w \in V_G \mid \{v, w\} \in E_G\}$. $M \subseteq V_G$ is called a *module (homogeneous set)* of G, if and only if for all $(v_1, v_2) \in M^2$: $N(v_1) - M = N(v_2) - M$, i.e. v_1 and v_2 have identical neighbourhoods outside M. $M \subseteq V_G$ is called a *trivial module*, if $|M| = 0$, $|M| = 1$, or $M = V_G$, see [CH94]. A graph G is called *prime* if every module of G is trivial. A module M is *maximal* if there is no non-trivial module N such that $M \subseteq N$. A module is called *strong* if it does not overlap with any other module.

While the set of modules of a graph G can be exponentially large, the set of strong modules is linear in the number of vertices. The inclusion order of the set of all strong modules defines a tree-structure which is denoted as *modular decomposition* T_G, see [MR84]. The root of T_G represents the graph G and the leaves of T_G correspond to the vertices of G. Every inner node, i.e. non-leaf node, w of T_G corresponds to an induced subgraph of G consisting of the leaves of T_G in subtree with root w, which is called the *representative graph* of w and is denoted by $G(w)$. Vertex set $V_{G(w)}$ is a strong module of G. For some inner node v of T_G, the *quotient graph* $G[v]$ is obtained by substituting in $G(v)$ every strong module, represented

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by some child of v in T_G, by a single vertex. For some inner node v of T_G, quotient graph $G[v]$ is either an independent set (v is denoted as *co-join node*), a clique (v is denoted as *join node*), or a prime graph (v is denoted as *prime node*).

For $U \subseteq V_G$, we define by $G[U]$ the subgraph of G induced by the vertices of U. For some graph G, we denote its edge complement by co-G. For a set of graphs $\mathcal{F}$, we denote by *$\mathcal{F}$-free graphs* the set of all graphs that do not contain a graph of $\mathcal{F}$ as an induced subgraph.

In Table 1 we show some special graphs to which we refer during the paper. A *hole* is a chordless cycle with at least five vertices. A *k-sun* is a chordal graph G on $2k$ vertices for some $k \geq 3$ whose vertex set can be partitioned into $V_G = U \cup W$ such that $U = \{u_0, \ldots, u_{k-1}\}$ and $W = \{w_0, \ldots, w_{k-1}\}$ is an independent set. Additionally vertex u_i is adjacent to vertex w_j if and only if $i = j$ or $i = j + 1 \mod k$. G is called a *sun* if it is a k-sun for some $k \geq 3$. If graph $G[U]$ is a clique, then G is called a *complete k-sun*.

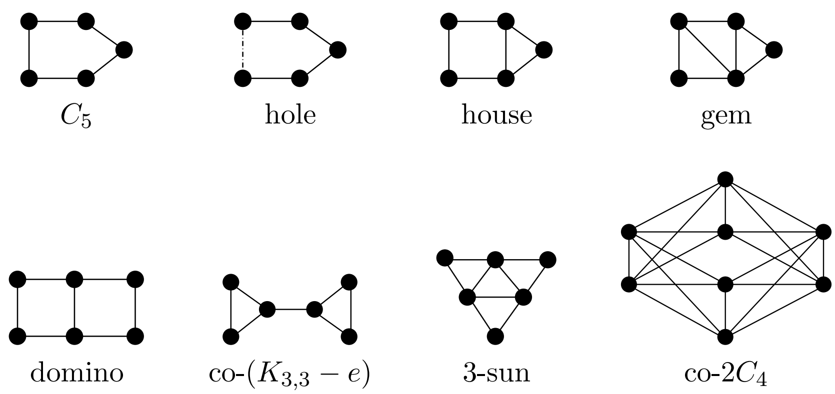

Table 1: Special graphs

2 Module-composed graphs

There are several graph classes which are defined by a sequence of one-vertex extentions of restricted form. Some well known examples are trees, co-graphs, and distance hereditary graphs, see [Rao07] for a survey. We next analyze a closely related but new concept.

Graph G is *module-composed*, if and only if there exists a linear ordering $\varphi : V_G \to [|V_G|]$, such that for every $2 \leq i \leq |V_G|$ the neighbourhood of vertex $\varphi^{-1}(i)$ in graph $G[\{\varphi^{-1}(1), \ldots, \varphi^{-1}(i-1)\}]$ forms a module. For some module-composed graph G, φ is called a *module-sequence* for G.

The definition of module-composed graphs was introduced [AGK+06] for computing connectivity ratings for vertices in special graph classes, see also [AKKW06]. We first recall the following easy but important lemma from [AGK+06].

Lemma 2.1 (Induced subgraph) *If a graph G is module-composed, then every induced subgraph of G is also module-composed.*

Given two module-sequences φ_1, φ_2 for two graphs G_1 and G_2, sequence $\varphi(v) = \varphi_1(v), v \in V_{G_1}$ and $\varphi(v) = \varphi_2(v) + |V_{G_1}|, v \in V_{G_2}$ is a possible module-sequence for the disjoint union of these two graphs.

Lemma 2.2 (Disjoint union) *For two module-composed graphs G_1, G_2, the disjoint union $G_1 \cup G_2$ is also module-composed.*

The following observation follows from Lemma 2.1 and the definition of module-composed graphs.

Lemma 2.3 *A graph G is module-composed, if and only if there exists a vertex $v \in V_G$ such that $N(v)$ is a module in graph $G[V_G - \{v\}]$ and graph $G[V_G - \{v\}]$ is module-composed.*

By Lemma 2.3 the following graphs (see Table 1) are not module-composed, since none of them contains a vertex v such that $N(v)$ is a module in graph $G[V_G - \{v\}]$:

C_n, $n \geq 5$ (i.e. holes), co-C_n, $n \geq 5$ (i.e. anti-holes), house, domino, co-$(K_{3,3} - e)$, 3-sun, co-$2C_4$.

The example of graph co-$2C_4$ shows that not every co-graph[1] is module-composed. Graph co-$2C_4$ can even be used to characterize those co-graphs which are module-composed.

Lemma 2.4 *Let G be a co-graph. The following conditions are equivalent.*

1. *G is module-composed.*
2. *G is (co-$2C_4$)-free.*

Proof If G is module-composed, then by Lemma 2.1 it obviously contains no co-$2C_4$ as induced subgraph.

Let G be (co-$2C_4$)-free co-graph. Then there exists a co-graph expression X defined by the three co-graph operations (single vertex $\bullet$, disjoint union $G_1 \cup G_2$ of two co-graphs G_1, G_2, join $G_1 \times G_2$ of two co-graphs G_1, G_2) for G. Any subexpression $\bullet$ and $G_1 \cup G_2$ are also feasible for a module-sequence.

Let $X' = X_1 \times X_2$ be a subexpression of X. Since the graph defined by X' contains no co-$2C_4$ as an induced subgraph either graph defined by X_1 or that by X_2 defines a subgraph of $K_1 \cup K_2$, i.e. the disjoint union of a clique on two vertices and a clique on one vertex. Let us assume that X_2 does so. This allows us to define a module decomposition for X as follows. We start with a module-sequence for X_1, which exists by induction, proceed with the vertices of K_2 and finish with vertex of graph K_1, which leads a module-sequence for graph defined by X. □

Co-graphs are exactly P_4-free graphs which implies our next corollary.

Corollary 2.5 *(co-$2C_4, P_4$)-free graphs are module-composed.*

Further it is known that trivially perfect[2] graphs are exactly (C_4, P_4)-free graphs [Gol78], which obviously form a subclass of (co-$2C_4, P_4$)-free graphs.

Corollary 2.6 *Trivially perfect graphs are module-composed.*

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[1] A *co-graph* is either a single vertex $\bullet$, the disjoint union $G_1 \cup G_2$ of two co-graphs G_1, G_2, or the join $G_1 \times G_2$ of two co-graphs G_1, G_2, which connects every vertex of G_1 with every vertex of G_2.

[2] A graph is *trivially perfect* if for every induced subgraph H of G, the size of the largest independent set in H equals the number of all maximal cliques in H.

Next we conclude results on super classes of module-composed graphs.

It is easy to see that the house, every hole and the domino are not module-composed. By a result shown in [Far83] each sun contains a complete sun as induced subgraph, which is obviously not module-composed. By Lemma 2.1 the next result follows.

Lemma 2.7 *Module-composed graphs are HHDS-free*[3].

Since HHDS-free graphs are perfect[4], the same holds true for module-composed graphs.

Corollary 2.8 *Module-composed graphs are perfect.*

Further, HHDS-free graphs are homogeneously orderable by the results shown in [BDN97], which implies the same for module-composed graphs.

Corollary 2.9 *Module-composed graphs are homogeneously orderable.*

Since the graph C_4 is module-composed but not chordal, we conclude that module-composed graphs are not chordal, but they are weakly chordal[5], since they are HHD-free[6] and HHD-free graphs are weakly chordal.

Corollary 2.10 *Module-composed graphs are weakly chordal.*

3 Algorithms for module-composed graphs

Next we give a polynomial time algorithm to recognize module-composed graphs. Our algorithm is based on Lemma 2.3. In order to find some vertex v that satisfies the conditions of Lemma 2.3, we use a modular decomposition [CH94] in our following Algorithm 3.1. A basic observation is that for every connected module-composed graph G vertex v is either a child or a grandchild of the root of T_G.

Algorithm 3.1

Input: Graph G
Output: Module-sequence $\varphi : V_G \to [|V_G|]$ or the answer NO

mod-com(G)
  if (G disconnected)
    for every connected component H of G: mod-com(H);
  else {
    construct T_G with root r;
    if (r is join node) {
      if ($\exists$ child v_l of r which is a leaf in T_G) {
        for every such child v_l of r $\{\varphi(v_l) = i++; G = G - \{v_l\}; \}$

[3](house,hole,domino,sun)-free

[4]A graph G is *perfect* if, for every induced subgraph H of G, the chromatic number of H is equal to the size of a maximum clique of H.

[5]A graph is *weakly chordal* if it does not contain any induced cycles of length greater than four or their complements.

[6](house,hole,domino)-free

(9) *mod-com(G);*}
(10) *else if* ($\exists$ *child* r_1 *of* r *labeled by co-join and a child* v_l *of* r_1 *which*
(11) *is a leaf in* T_G) {
(12) *for every such vertex* v_l $\{\varphi(v_l) = i{+}{+}; G = G - \{v_l\};\}$
(13) *mod-com(G);* }
(14) }
(15) *else if* (r *is prime node*) {
(16) *if* ($\exists$ *child* v_1 *of* r *which is a leaf in* T_G *and corresponds to a vertex*
(17) *of degree 1 in quotient graph* $G[r]$) {
(18) *for every such child* v_1 *of* r $\{\varphi(v_1) = i{+}{+}; G = G - \{v_1\};\}$
(19) *mod-com(G);*}
(20) *else if* ($\exists$ *child* r_1 *of* r *labeled by co-join and corresponds to a vertex*
(21) *of degree 1 in quotient graph* $G[r]$ *and a child* v_1 *of* r_1 *which is a*
(22) *leaf in* T_G) {
(23) *for every such vertex* v_1 $\{\varphi(v_1) = i{+}{+}; G = G - \{v_1\};\}$
(24) *mod-com(G);* }
(25) }
(26) *else*
(27) *return NO;*
(28) }

The construction of the modular decomposition T_G in Line (5) of Algorithm 3.1 can be realized in time $O(|V_G| + |E_G|)$ by [CH94, MS99].

Theorem 3.2 *Given a graph G, one can decide in time $O(|V_G| \cdot (|V_G| + |E_G|))$ whether G is module-composed, and in the case of a positive answer, constructs a module-sequence.*

Since module-composed graphs are HHD-free, we conclude by the results shown in [JO88] the following theorem.

Theorem 3.3 *For every module-composed graph which is given together with a module-sequence the size of a largest independent set, the size of a largest clique, the chromatic number and the minimum number of cliques covering the graph can be computed in linear time.*

4 Independent module-composed graphs

Next we want to characterize module-composed graphs for a restricted case.

A graph G is *independent module-composed*, if and only if there exists a linear ordering $\varphi : V_G \to [|V_G|]$, such that for every $2 \leq i \leq |V_G|$ the neighbourhood of vertex $\varphi^{-1}(i)$ in graph $G[\{\varphi^{-1}(1), \ldots, \varphi^{-1}(i-1)\}]$ forms a module which is an independent set.

It is easy to see that independent module-composed graphs do not contain any of the graphs of Table 1 as induced subgraph.

Lemma 4.1 *Independent module-composed graphs are HHDG-free*[7]*.*

[7] (house,hole,domino,gem)-free

HHDG-free are also known as distance hereditary graphs [HM90, BM86]. Examples for distance hereditary graphs are co-graphs and trees. For the case of bipartite graphs[8], the notion module-composed even is equivalent to the notion of distance hereditary.

Theorem 4.2 ([AGK$^+$06]) *Let G a bipartite graph. The following conditions are equivalent.*

1. *G is module-composed.*
2. *G is domino and hole free.*
3. *G is distance hereditary.*
4. *G is $(6,2)$-chordal[9].*

For general graphs Theorem 4.2 does not hold true, since there are module-composed graphs which are not distance hereditary, e.g. the gem and there are distance hereditary graph which are not module-composed, e.g. the co-$(K_{3,3} - e)$.

The problem to decide whether a given graph is bipartite distance hereditary and to construct a corresponding pruning sequence can be done in linear time by the well known characterization for bipartite graphs as 2-colorable graphs and the linear time recognition algorithms for distance hereditary graphs shown in [HM90, BM86]. By Theorem 4.2, this immediately implies a linear time algorithms for recognizing independent module-composed graphs. A corresponding module-sequence can be constructed in linear time from a pruning sequence as shown in [AGK$^+$06]. Since both known linear time recognition algorithms for distance hereditary graphs shown in [HM90, BM86] are based on the fact that the neighbourhood of every vertex in a distance hereditary graph is a co-graph and additional conditions, both algorithms are not simple.

In [JO88] it is shown that for HHD-free graphs every Lex-BFS (Lexicographic Breadth First Search) ordering is a semi perfect elimination ordering, i.e. every vertex $\varphi^{-1}(i)$ is no midpoint of an induced P_4 in graph $G[\{\varphi^{-1}(1), \ldots, \varphi^{-1}(i-1)\}]$. In the case of bipartite graphs this ordering obviously is even an independent module-sequence.

Theorem 4.3 *Given an independent module-composed graph G, every Lex-BFS ordering constructs in time $O(|V_G| + |E_G|)$ an independent module-sequence for G.*

To decide whether a given graph is bipartite distance hereditary can be done by Corollary 5 shown in [BM86] using the fundamental search strategy of BFS (Breadth First Search) which produces a classification of the vertices into levels, with respect to a start vertex u. Level i is the set of vertices with distance i to vertex u and is denoted by $N_i(u)$.

Theorem 4.4 (Corollary 5 of [BM86]) *Let G be a connected graph and let u be a vertex of G. Then G is bipartite distance hereditary if and only if all levels $N_k(u)$ are edgeless, and for every vertices v,w in $N_k(u)$ and neighbours x and y of v in $N_{k-1}(u)$, we have $N(x) \cap N_{k-2}(u) = N(y) \cap N_{k-2}(u)$, and further $N(v) \cap N_{k-1}(u)$ and $N(w) \cap N_{k-1}(u)$ are either disjoint or one is contained in the other.*

[8] A graph is bipartite if it is C_{2n+1}-free, for $n \geq 1$.

[9] A graph is (k,l)-chordal if each cycle of length at least k has at least l chords.

A BFS starting at a vertex u can compute the level sets $N_k(u)$ in time $O(|V_G|+|E_G|)$ and using these levels, the conditions of Corollary 5 of [BM86] can be verified in the same time.

A BFS numbering φ of the vertices with respect to some vertex u can be used to obtain a module-sequence φ_1 as follows. We start with $\varphi_1(v) = \varphi(v)$, $\forall v \in V_G$. For the first $|N_0(u)|+|N_1(u)|$ vertices we obviously can choose $\varphi_1(v) = \varphi(v)$. For the vertices of $w \in N_k(u)$, $k \geq 2$, we know that their neighbours in set $N_{k-1}(u)$ are modules which can be ordered by a series of inclusions $N^1 \subseteq N^2 \subseteq \ldots \subseteq N^j$. We rearrange the order of the vertices in $N_k(u)$ with respect to φ_1 such that for every such series of inclusions $\varphi_1(w_1) < \varphi_1(w_2)$ if and only if $N_{k-1}(u) \cap N(w_1) \supseteq N_{k-1}(u) \cap N(w_2)$. This obviously leads a module-sequence for graph G if G is bipartite distance hereditary.

Theorem 4.5 *Given a graph G, one can decide using BFS in time $O(|V_G| + |E_G|)$ whether G is independent module-composed, and in the case of a positive answer, construct a module-sequence.*

On bipartite distance hereditary graphs, and so on independent module-composed graphs, the path-partition problem [YC98], hamiltonian circuit and path problem [MN93], and the computation of shapley value ratings [AGK$^+$06] can be solved in polynomial time.

It is well known that distance hereditary graphs and thus independent module-composed graphs have clique-width at most 3 [GR00]. This implies that all graph properties which are expressible in monadic second order logic with quantifications over vertices and vertex sets (MSO_1-logic) are decidable in linear time on independent module-composed graphs [CMR00]. Some of these problems are partition into k independent sets or cliques, k-dominating set, k-achromatic number, for every fixed integer k.

Furthermore, there are a lot of NP-complete graph problems which are not expressible in MSO_1-logic like chromatic number, partition problems, vertex disjoint paths, and bounded degree subgraph problems but which can also be solved in polynomial time on clique-width bounded graphs and thus on bipartite distance hereditary graphs [EGW01, GW06].

Note that general module-composed graphs are of unbounded clique-width. For example every graph which can be constructed from a single vertex by a sequence of one vertex extentions by a domination vertex[10] or a pendant vertex[11] is obviously module-composed. But the set of all such defined graphs have unbounded clique-width [Rao07].

5 Graph class inclusions

In Table 2 we summarize the relation of module-composed graphs and related graph classes. For the definition and relations of special graph classes we refer to the survey of Brandstädt et al. [BLS99].

References

[AGK$^+$06] M. Abraham, F. Gurski, A. Krumnack, R. Kötter, and E. Wanke. A connectivity rating for vertices in networks. submitted, 2006.

[10] A vertex $v \in V_G$ is a *dominating vertex* of G, if it is adjacent to all other vertices in G.

[11] A vertex $v \in V_G$ of degree one is called a *pendant vertex* of G.

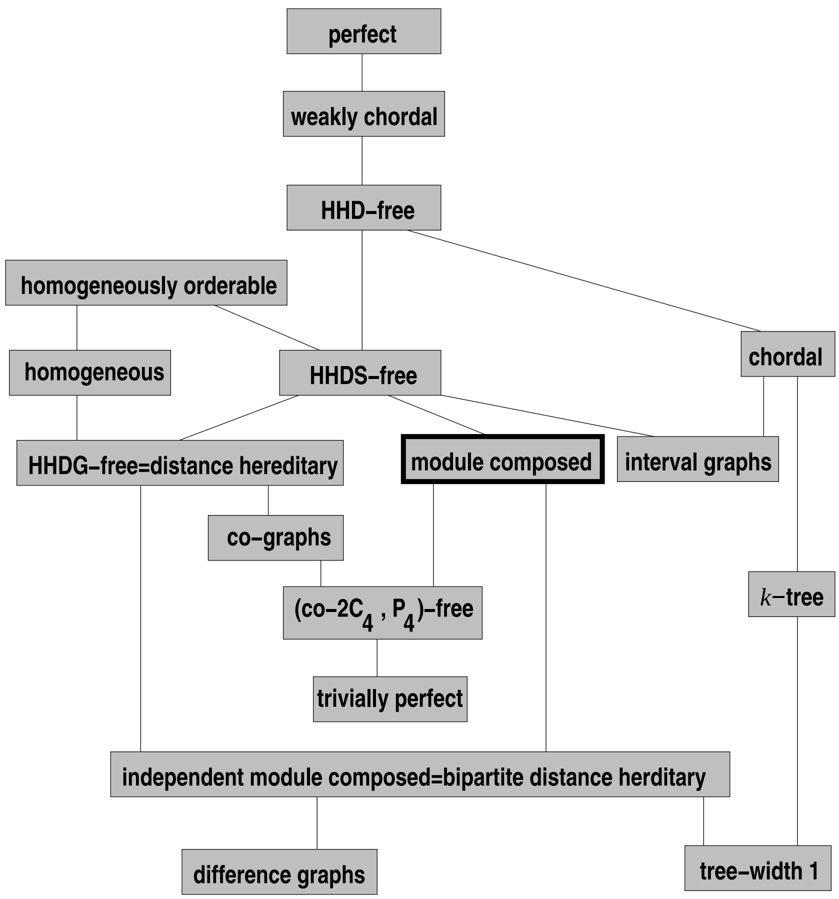

Table 2: Inclusion of special graph classes

[AKKW06] M. Abraham, R. Kötter, A. Krumnack, and E. Wanke. A connectivity rating for vertices in networks. In *Proceedings of the 4th IFIP International Conference on Theoretical Computer Science-TCS*, pages 283–298. Springer, 2006.

[BDN97] A. Brandstädt, F. F. Dragan, and F. Nicolai. Homogeneously orderable graphs. *Theoretical Computer Science*, 172:209–232, 1997.

[BLS99] A. Brandstädt, V.B. Le, and J.P. Spinrad. *Graph Classes: A Survey.* SIAM Monographs on Discrete Mathematics and Applications. SIAM, Philadelphia, 1999.

[BM86] H.-J. Bandelt and H.M. Mulder. Distance-hereditary graphs. *Journal of Combinatorial Theory, Series B*, 41:182–208, 1986.

[CH94] A. Cournier and M. Habib. A new linear time algorithm for modular decomposition. In *Procedings of CAAP*, volume 787 of *LNCS*, pages 68–84. Springer, 1994.

[CMR00] B. Courcelle, J.A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory of Computing Systems*, 33(2):125–150, 2000.

[EGW01] W. Espelage, F. Gurski, and E. Wanke. How to solve NP-hard graph problems on clique-width bounded graphs in polynomial time. In *Proceedings of Graph-Theoretical Concepts in Computer Science*, volume 2204 of *LNCS*, pages 117–128. Springer, 2001.

[Far83] M. Farber. Characterizations of strongly chordal graphs. *Discrete Mathematics*, 43:173–189, 1983.

[Gol78] M.C. Golumbic. Trivially perfect graphs. *Discrete Mathematics*, 24:105–107, 1978.

[GR00] M.C. Golumbic and U. Rotics. On the clique-width of some perfect graph classes. *International Journal of Foundations of Computer Science*, 11(3):423–443, 2000.

[GW06] F. Gurski and E. Wanke. Vertex disjoint paths on clique-width bounded graphs. *Theoretical Computer Science*, 359(1-3):188–199, 2006.

[HM90] P.L. Hammer and F. Maffray. Completely separable graphs. *Discrete Applied Mathematics*, 27:85–99, 1990.

[JO88] B. Jamison and S. Olariu. On the semi-perfect elimination. *Advances in applied mathematics*, 9:364–376, 1988.

[MN93] H. Müller and F. Nicolai. Polynomial time algorithms for hamiltonian problems on bipartite distance-hereditary graphs. *Information Processing Letters*, 46(5):225–230, 1993.

[MR84] R.H. Möhring and F.J. Radermacher. Substitution decomposition for discrete structures and connections with combinatorial optimization. *Annals of Discrete Mathematics*, 19:257–365, 1984.

[MS99] R.M. McConnell and J. Spinrad. Modular decomposition and transitive orientation. *Discrete Mathematics*, 201(1-3):189–241, 1999.

[Rao07] M. Rao. Clique-width of graphs defined by one-vertex extensions. Manuscript, 2007.

[YC98] H.-G. Yeh and G.J. Chang. The path-partition problem in bipartite distance-hereditary graphs. *Taiwanese Journal of Mathematics*, 2(3):353–360, 1998.